

One hundred and ten presents were announced as having been received since the last meeting, including, amongst others :—

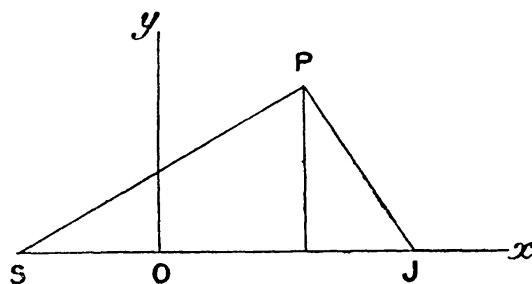
A. Auwers, *Mittlere Örter von 570 Sternen aus den auf der Sternwarte Greenwich unter Direction von Pond in den Jahren 1811–1819 angestellten Beobachtungen abgeleitet* (presented by the author) ; Professional papers of the Survey of India Department, No. 5, containing Major S. G. Burrard's paper on the attraction of the Himalaya Mountains (presented by the Survey Department).

On Periodic Orbits in the Restricted Problem of Three Bodies.

By E. T. Whittaker, M.A.

1. *Introduction.*—In a paper published in the *Monthly Notices*, vol. lxii. pp. 186–193, two properties of periodic orbits were communicated. The periodic orbits discussed in that paper were the closed curves which can be described by a body moving under the influence of any number of centres of force. It was supposed that these centres of force are fixed, so that the energy of the body can be partitioned into kinetic energy, which is a homogeneous quadratic function of the components of velocity, and potential energy, which does not involve the components of velocity or the time at all.

The application of periodic orbits, which is most important from the astronomical point of view, is the application to the restricted problem of three bodies (the orbits of Hill, Poincaré, and Darwin). This problem consists in the determination of the motion of a small planet P under the attraction of the Sun S and a large planet J, it being assumed that the motion of S and J is purely circular, and that the whole motion takes place in one plane. In this problem the conditions are not so simple as in the class of problems studied in the previous paper, since S and J are not fixed but *moving* centres of force, and the theorems here stated cannot be applied.



The object of the present note is to show that the methods of the previous paper can nevertheless be extended to the restricted problem of three bodies, and that two results can thus be

obtained which are analogous to, though differing from, those given in the previous paper.

2. *Formation of the equation of Least Action.*—Of the attracting bodies let S be of mass m_1 , and J of mass m_2 . Let O be their centre of gravity, which we can regard as fixed. Let x and y be the co-ordinates of P, when the point O is taken as origin, with the moving line OJ as axis of x , and a line perpendicular to it as axis of y . Let $\dot{x}$ be the angular velocity of OJ, and let

$$F(x, y) = \frac{m_1}{SP} + \frac{m_2}{JP}.$$

Then the differential equations of motion of the problem are

$$\begin{cases} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 0, \end{cases}$$

where the Lagrangian function L is given by the equation

$$L = \frac{1}{2} \{ (\dot{x} - ny)^2 + (\dot{y} + nx)^2 \} + F(x, y),$$

and $\dot{x}, \dot{y}$ denote $\frac{dx}{dt}, \frac{dy}{dt}$ respectively.

The corresponding integral of energy is

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{n^2}{2}(x^2 + y^2) - F(x, y) = h,$$

where h is the constant of energy.

Now write,

$$y' = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}},$$

and write

$$\Omega = \frac{1}{2} \{ (\dot{x} - ny)^2 + (y'\dot{x} + nx)^2 \} + F(x, y),$$

so that it is equal to L, but is expressed differently,

Now the integral of energy can be written

$$\dot{x} \frac{\partial \Omega}{\partial \dot{x}} - \Omega = h,$$

and this equation can be regarded as giving $\dot{x}$ in terms of x, y, y' , and h . Differentiating it on this supposition, we have

$$\dot{x} \frac{\partial^2 \Omega}{\partial \dot{x}^2} \frac{\partial \dot{x}}{\partial y'} + \dot{x} \frac{\partial^2 \Omega}{\partial \dot{x} \partial y'} - \frac{\partial \Omega}{\partial y'} = 0$$

and

$$\dot{x} \frac{\partial^2 \Omega}{\partial \dot{x}^2} \frac{\partial \dot{x}}{\partial y} + \dot{x} \frac{\partial^2 \Omega}{\partial \dot{x} \partial y} - \frac{\partial \Omega}{\partial y} = 0.$$

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Now let $L' = \frac{\partial \Omega}{\partial \dot{x}}$, where L' is supposed expressed in terms of x , y , and y' , so that

$$L' = n(xy' - y) + (1 + y'^2)^{\frac{1}{2}} \{n^2(x^2 + y^2) + 2F(x, y) + 2h\}^{\frac{1}{2}}.$$

Then the two last equations become

$$\dot{x} \frac{\partial L'}{\partial y'} - \frac{\partial \Omega}{\partial y'} = 0$$

and

$$\dot{x} \frac{\partial L'}{\partial y} - \frac{\partial \Omega}{\partial y} = 0,$$

or

$$\frac{\partial L'}{\partial y'} = \frac{\partial L}{\partial \dot{y}}$$

and

$$\frac{\partial L'}{\partial y} = \frac{1}{\dot{x}} \frac{\partial L}{\partial y}.$$

The equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 0$$

becomes therefore

$$\frac{d}{dx} \left(\frac{\partial L'}{\partial y'} \right) - \frac{\partial L'}{\partial y} = 0,$$

and this single equation, which connects y and x and does not involve t , is the differential equation of those orbits for which the constant of energy is h .

To this equation there corresponds a statement of the principle of least action, namely that the integral

$$\int L' dx$$

has a value which is stationary for the actual trajectory between two given terminal points, as compared with neighbouring curves joining the same two terminal points.

3. *A criterion for the discovery of periodic orbits.*—Now consider any simple closed curve in the plane of xy ; and let another simple closed curve C' be drawn, enclosing C and differing only slightly from it. We may regard C' as defined by an equation of the form

$$\delta p = \text{a function of } \phi,$$

where δp is the normal distance between the curves C and C' , and ϕ is the inclination of this normal distance to the axis of x .

Then if I be the value of the integral

$$\int L' dx$$

when the integration is taken round the curve C , and if $I + \delta I$ denote the value of the same integral when the integration is taken round the curve C' (so that the symbol δ denotes an increment obtained in passing from C to C'), we find, by a method similar to that used in the previous paper, that

$$\delta I = \int \delta p \cdot K \cdot \{(dx)^2 + (dy)^2\}^{\frac{1}{2}},$$

where, writing

$$\{h + F(x, y) + \frac{1}{2}n^2(x^2 + y^2)\}^{\frac{1}{2}} = G(x, y),$$

we have

$$K = \frac{G(x, y)}{\phi} + \cos \phi \frac{\partial G(x, y)}{\partial x} + \sin \phi \frac{\partial G(x, y)}{\partial y} + \frac{n}{\sqrt{2}}.$$

This equation shows that if the quantity K is positive at all points of C , then δI is positive, and, so the quantity I has its value increased when any curve surrounding C and adjacent to C is taken instead of C as the path of integration.

Now, suppose that another simple closed curve D can be drawn enclosing C , and such that at all points of D the quantity K is negative. Then in the same way it can be shown that the quantity I is increased when any simple closed curve D' , enclosed by D and adjacent to D , is taken instead of D as the path of integration.

It follows, therefore, by reasoning similar to that used in the previous paper, that, among the aggregate of simple closed curves situated in the ring-shaped space bounded by C and D , there exists a curve L , for which the value of the integral I is greater than for all the other curves of the aggregate. The curve L thus furnishes a stationary value of the integral of least action, and so is an orbit in the restricted problem of three bodies. We thus obtain theorem

If one closed curve be enclosed by another closed curve, and if the quantity K be positive at all points of the inner curve and negative at all points of the outer curve, then in the ring-shaped space between the two curves there exists a periodic orbit of the restricted problem of three bodies, having the given arbitrary constant h as its constant of energy.

As the quantity K can be immediately calculated for every point on any given curve, this result furnishes a means of detecting the presence of periodic orbits. It may be regarded as analogous to those methods of finding the position of roots of an algebraic equation, which depend on considerations relating to the sign of some expressions connected with the equation.

4. *An integral connected with the periodic orbits.*—Consider now the integral

$$\frac{1}{2\pi} \iint \left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log \{h + F(x, y) + \frac{1}{2}n^2(x^2 + y^2)\} \right] dx dy,$$

where the integration is taken over the interior of any periodic orbit C in the restricted problem of three bodies. As the integrand is infinite at the points S and J we shall first suppose that, if S and J are in the interior of the orbit, small circles γ and γ' are drawn around them. If S and J are not enclosed by the periodic orbit the circles γ and γ' will be non-existent; but in order to secure the greatest generality we shall assume that they exist.

Then, if S denote the region which is contained within C but is exterior to γ and γ' , we have (by Green's theorem on the conversion of integrals taken over an area into integrals taken round the boundary of the area) the equality

$$\begin{aligned} & \frac{1}{2\pi} \iint_S \left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log \{h + F(x, y) + \frac{1}{2}n^2(x^2 + y^2)\} \right] dx dy \\ &= \frac{1}{2\pi} \left[\int_C - \int_\gamma - \int_{\gamma'} \right] \left[\begin{aligned} & dy \frac{\partial}{\partial x} \log \{h + F(x, y) + \frac{1}{2}n^2(x^2 + y^2)\} \\ & - dx \frac{\partial}{\partial y} \log \{h + F(x, y) + \frac{1}{2}n^2(x^2 + y^2)\} \end{aligned} \right] \end{aligned}$$

the integrals round γ and γ' being inserted only when γ and γ' exist.

But the integral previously denoted by δI must be zero when the integration is taken along an arc of an actual orbit; and this (since the quantity δp which occurs in δI is arbitrary) shows that the equation

$$K = 0$$

$$\text{or } \frac{1}{\rho} + \cos \phi \frac{\partial \log G(x, y)}{\partial x} + \sin \phi \frac{\partial \log G(x, y)}{\partial y} + \frac{n}{\sqrt{2G(x, y)}} = 0$$

is satisfied at every point of the orbit; that is, the equation

$$\begin{aligned} & 2 \{dx^2 + dy^2\}^{-\frac{1}{2}} d\phi + \left(\cos \phi \frac{\partial}{\partial x} + \sin \phi \frac{\partial}{\partial y} \right) \log \{h + F(x, y) \\ & + \frac{1}{2}n^2(x^2 + y^2)\} + \sqrt{2n} \{h + F(x, y) + \frac{1}{2}n^2(x^2 + y^2)\}^{\frac{1}{2}} = 0 \end{aligned}$$

or

$$\begin{aligned} & 2d\phi + dy \frac{\partial}{\partial x} \log \{h + F(x, y) + \frac{1}{2}n^2(x^2 + y^2)\} \\ & - dx \cdot \frac{\partial}{\partial y} \log \{h + F(x, y) + \frac{1}{2}n^2(x^2 + y^2)\} + 2n dt = 0 \end{aligned}$$

is satisfied at every point of the curve C. We have therefore

$$\begin{aligned} \frac{1}{2\pi} \int_C \left(dy \frac{\partial}{\partial x} - dx \frac{\partial}{\partial y} \right) \log \{ h + F(x, y) + \frac{1}{2} n^2 (x^2 + y^2) \} \\ = - \frac{1}{2\pi} \int_C (2d\phi + 2ndt) \\ = -2 - \frac{\Gamma}{\pi}, \end{aligned}$$

where Γ is the angle through which the whole system of S, J, and P has turned in space during one period of the periodic orbit.

We have next to consider the integrals which arise from the curves γ and γ' . We shall take first the integral arising from the curve γ . We shall suppose that the centre of γ is at S, and that its radius is a small quantity f .

Then in the neighbourhood of S we have

$$h + F(x, y) + \frac{n^2}{2} (x^2 + y^2) = \frac{m_1}{SP} + \text{terms finite in the neighbourhood};$$

$$\begin{aligned} \text{thus} \quad \left(dy \frac{\partial}{\partial x} - dx \frac{\partial}{\partial y} \right) \log \left\{ h + F(x, y) + \frac{n^2}{2} (x^2 + y^2) \right\} \\ = -dy \frac{OS + x}{SP^2} + dx \cdot \frac{y}{SP^2} + \text{less important terms.} \end{aligned}$$

Therefore

$$\begin{aligned} - \frac{1}{2\pi} \int_{\gamma} \left(dy \frac{\partial}{\partial x} - dx \frac{\partial}{\partial y} \right) \log \left\{ h + F(x, y) + \frac{n^2}{2} (x^2 + y^2) \right\} \\ = \frac{1}{2\pi\rho^2} \int_{\gamma} \{ (OS + x)dy - ydx \} + \text{infinitesimal terms} \\ = \frac{1}{2\pi\rho^2} \times \text{twice the area of } \gamma \\ = 1. \end{aligned}$$

Similarly the integral arising from γ' has the value unity.

This result is clearly independent of the radius of the small circles γ and γ' , provided these are infinitesimal; we are thus entitled to speak of the integral of the quantity

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log \left\{ h + F(x, y) + \frac{n^2}{2} (x^2 + y^2) \right\}$$

taken over the interior of the periodic orbit, since the arbitrary

element of choice in the circles γ and γ' does not affect the value of the integral. We thus obtain the result—

The integral

$$\frac{1}{2\pi} \iint \left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log \{h + F(x, y) + \frac{1}{2}n^2(x^2 + y^2)\} \right] dx dy,$$

where the integration is taken over the interior of any periodic orbit in the restricted problem of three bodies, has the value

$$-\frac{\Gamma}{\pi}, \text{ or } -1 - \frac{\Gamma}{\pi}, \text{ or } -2 - \frac{\Gamma}{\pi},$$

according as the periodic orbit encloses both, one, or none of the bodies S and J; where Γ is the angle through which the whole system composed of S, J, and P turns in space during one period of the periodic orbit.

1902 January 24.

*On the Images formed by a Parabolic Mirror. First Paper :
The Geometrical Theory.* By H. C. Plummer, M.A.

1. *Introduction.*—As is well known the staple work of the University Observatory at Oxford during the past few years has been the execution of its share of the Astrographic Chart. At the same time lines of investigation suggested by this work have not been neglected, and much has been done in the discussion of the systematic errors to which photographs are liable. As an example of such researches reference may be made to Professor Turner's papers on the optical distortion of the photographic doublet (*Monthly Notices*, vol. lix. p. 438).

Soon after I became associated as assistant with the observatory Professor Turner suggested that, as I had already paid some attention to the forms of the images produced by a reflecting telescope (*A. J.*, No. 435) I should extend and as far as possible complete this study. Accordingly I undertook the direct comparison of a plate kindly lent by Dr. Isaac Roberts with one of our own astrographic plates. This task is proceeding and will, I hope, shortly be finished. In the meantime a comparison of my own earlier paper with various other theoretical papers revealed apparent discrepancies which it is desirable to remove by a thorough revision of this part of the subject. The results of this investigation are independent of the actual comparison of the two plates, and it seems convenient therefore to publish them separately.

2. My earlier paper, "On the Star Image formed by a Parabolic Mirror," was published in the *Astronomical Journal*,